

# Cognitive constraints, contraction consistency, and the satisficing criterion

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## Abstract

A theory of decision making is proposed that offers an axiomatic basis for the notion of “satisficing” postulated by Herbert Simon. The theory relaxes the standard assumption that the decision maker always fully perceives his preferences among the available alternatives, requiring instead that his ability to perceive any given preference be decreasing with respect to the complexity of the choice problem at hand. When complexity is aligned with set inclusion, this exercise is shown to be equivalent to abandoning the contraction consistency axiom of classical choice theory.

*JEL classification:* D01; D11; D80

*Keywords:* Choice function; Perception; Revealed preference; Threshold

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## 1. Introduction

Now more than fifty years ago, Herbert Simon published the first [24] of several early articles challenging the models of decision making then dominant in economic analysis. “[T]he task,” he wrote [p. 99],

is to replace the global rationality of economic man with a kind of rational behavior that is compatible with the access to information and the computational capacities that are actually possessed by organisms, including man, in the kinds of environments in which such organisms exist.

As an example of such compatible behavior Simon proposed the concept of “satisficing,”<sup>1</sup> which he proceeded to define [25, v. 3, p. 295] as “choos[ing] an alternative that meets or exceeds specified criteria, but that is not guaranteed to be either unique or in any sense the best.” This proposal can be implemented by replacing the standard *maximizing criterion*  $f(x) \geq \max f[A]$  for an alternative  $x$  to be selectable from a menu  $A$  with the *satisficing criterion*  $f(x) \geq \theta(A)$ , where the functions  $f$  and  $\theta$  return, respectively, utility values and thresholds for acceptability. And it then follows that satisficing behavior includes

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<sup>1</sup>Simon [25, v. 2, p. 415] identifies this word as Scottish in origin, while the *Oxford English Dictionary* finds its earliest recorded use in the Swiss theologian Henry Bullinger’s [7] comment (presumably about the Romans) “[t]hat their founders were nourished by suckyng of a wolfe: so haue all that people wolues mindes, neuer satisfised with bloud, euer greedy of dominion and hungryng after riches. . . .”

$wxyz \mapsto xyz$ $xRw \ \& \ yRw \ \& \ zRw$ $\theta(wxyz) = 1$					
$wxy \mapsto xy$ $xRw \ \& \ yRw$ $\theta(wxy) = 1$	$wxz \mapsto z$ $xRw \ \& \ zRw \ \& \ zRx$ $\theta(wxz) = 3$		$wyz \mapsto yz$ $yRw \ \& \ zRw$ $\theta(wyz) = 2$	$xyz \mapsto z$ $zRx \ \& \ zRy$ $\theta(xyz) = 3$	
$wx \mapsto x$ $xRw$ $\theta(wx) = 1$	$wy \mapsto y$ $yRw$ $\theta(wy) = 2$	$wz \mapsto z$ $zRw$ $\theta(wz) = 3$	$xy \mapsto xy$ $\theta(xy) = 1$	$xz \mapsto z$ $zRx$ $\theta(xz) = 3$	$yz \mapsto z$ $zRy$ $\theta(yz) = 3$
$w \mapsto w$ $\theta(w) = 0 = f(w)$		$x \mapsto x$ $\theta(x) = 1 = f(x)$		$y \mapsto y$ $\theta(y) = 2 = f(y)$	
				$z \mapsto z$ $\theta(z) = 3 = f(z)$	

Figure 1: An illustrative example. A choice problem is a nonempty subset of  $X = wxyz$ ; the binary relation  $R$  indicates strict preference; the function  $f$  assigns utility values to alternatives; and the function  $\theta$  assigns utility thresholds to menus. Within the cells are displayed the mapping from menus to subsets of acceptable alternatives (e.g.,  $wxy \mapsto xy$ ), the preferences perceived in the corresponding choice problems (e.g.,  $xRw$  and  $yRw$  in problem  $wxy$ ), and the relevant thresholds (e.g.,  $\theta(wxy) = 1$ ).

maximizing behavior as a special case, so that any axiomatic basis for the former must be logically weaker than the classical axiomatizations of the latter.

This paper will suggest a rationale for satisficing behavior based on two themes of Simon's work: *cognitive constraints* and *environmental complexity*. Specifically, we shall consider the possibility that a decision maker's cognitive limitations may to some degree prevent him from perceiving his own preferences among the available alternatives, and shall assume that the magnitude of this effect is increasing with respect to the complexity of the choice problem at hand. The further assumption that complexity is aligned with set inclusion will then lead to our "nestedness" hypothesis that any (strict) preference perceived in a given problem  $B$  is also perceived in each simpler problem  $A \subset B$  in which it is relevant. And our principal finding will be that formalizing Herbert Simon's critique in this way leads us to abandon the contraction consistency axiom of classical choice theory (which requires any alternative both acceptable in  $B$  and available in  $A \subset B$  also to be acceptable in  $A$ ).

The main features of our theory are illustrated by the example depicted in Figure 1, in which one cell is assigned to each choice problem drawn from the universal set  $X = wxyz$ . (Note the multiplicative notation for enumerated sets.) Here the top entry in each cell shows the subset of acceptable alternatives associated with the problem in question; e.g., any element of  $xy$  is acceptable when the menu is  $wxy$ . The middle entry lists the (strict) preferences that are perceived in the corresponding situation and with respect to which the acceptable alternatives are maximal; e.g., the perceived preference  $zRx$  causes  $x$  to be *unacceptable* in problem  $wxz$ . And the bottom entry supplies the applicable utility threshold for acceptability; e.g., the acceptable alternatives on the menu  $wyz$  are those with utility values no smaller than  $\theta(wyz) = 2$ . Moreover, the pattern of perceived preferences displayed in this example is nested; e.g., the preference  $zRx$  is perceived in the problem  $xyz \supset xz$  and hence also in the problem  $xz$ . And finally, the decision maker's behavior violates contraction consistency; e.g., alternative  $x$  is deemed acceptable in problem  $wxyz$  but not in problem  $wxz \subset wxyz$ .

The purpose of this paper is to establish certain formal relationships among the three different types of constructs that appear in Figure 1. In Section 2, we demonstrate that choice behavior is governed by a nested system of perceived preferences with the standard (weak) ordering properties if and only if it satisfies an axiom of *expansion* consistency, thus relating cognitive constraints to failures of *contraction* consistency. In Section 3, we then show that this same expansion consistency condition can be used to axiomatize a special case of the satisficing criterion introduced above. Section 4 establishes that the link between cognitive constraints and failures of contraction consistency is independent of the ordering properties needed for the satisficing result. And the concluding Section 5 discusses related work as well as some conceptual issues raised by our investigations.

Appendices A–B contain essential material that will be of interest primarily to specialists. Appendix C provides selected proofs.

## 2. Choice, preference, and cognition

### 2.1. Preliminaries

Given a nonempty set  $X$ , the set of all *menus* drawn from  $X$  is  $\mathcal{A} = \{A \subset X : A \neq \emptyset\}$ . Any particular  $A \in \mathcal{A}$  then represents a slate of mutually-exclusive alternatives, while  $X$  amounts to a full catalog of the options potentially available. A *choice function* on  $X$  is a mapping  $C : \mathcal{A} \rightarrow \mathcal{A}$  with the property that  $\forall A$  we have  $C(A) \subset A$ .<sup>2</sup> Its value  $C(A)$  at  $A$  is referred to as the *choice set* associated with this menu, and can be interpreted as the collection of alternatives whose selection from  $A$  cannot be ruled out on the basis of whatever assumptions have been made about the process of decision making.

A *binary relation*  $R$  on  $X$  is a subset of  $X \times X$ , though we ordinarily abbreviate  $\langle x, y \rangle \in R$  as  $xRy$ . Such a relation is said to be a *partial order* when it is both acyclic ( $x_1Rx_2R \cdots Rx_n$  only if  $x_1 \neq x_n$ ) and transitive ( $xRyRz$  only if  $xRz$ ); a *weak order* when it is a negatively transitive ( $xRz$  only if either  $xRy$  or  $yRz$ ) partial order; and a *linear order* when it is a weakly connected ( $x \neq y$  only if either  $xRy$  or  $yRx$ ) weak order.

A relation  $R$  is said to *generate* a choice function  $C$  when each choice set contains those and only those available alternatives that are maximal with respect to  $R$ ; i.e., when  $\forall A$  we have  $C(A) = \{x \in A : \forall y \in A \text{ we have } \neg[yRx]\}$ .

### 2.2. Existence versus perceivability of preferences

Neoclassical economic theory rests on the assumption that the choice function encoding the behavior of each autonomous agent is generated by a preference relation with the properties of a weak order. Moreover, Samuelson [19] and Arrow [4] have shown that the observable implications of this assumption are captured in a condition generally known as the Weak Axiom of Revealed Preference.

**Condition 1 (Weak Axiom)**  $\forall x, y \in A \cap B$  such that both  $x \in C(A)$  and  $y \in C(B)$ , we have  $x \in C(B)$ .

**Theorem 1 (Arrow)** *A choice function is generated by a weak order if and only if it satisfies the Weak Axiom.*

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<sup>2</sup>Note that the domain of  $C$  includes each set of alternatives that could conceivably be assembled. This requirement that choice functions be “fully extended” is discussed at length by Arrow [4, p. 122], Herzberger [14, p. 192], and Sen [21, p. 245 ff.].

Our interest at present lies in relaxing the assumption of preference-maximal choice so as to allow cognitive constraints and environmental complexity to influence behavior. To this end, let us consider a decision maker who possesses a (strict) preference relation  $R$  of the usual sort, but who cannot be relied upon to perceive any particular preference  $xRy$  in each choice problem  $A \supset xy$  in which it is relevant. In this case, asserting the *existence* of the preference in question no longer allows us to conclude — as in the standard case — that  $y$  will never be chosen when  $x$  is available. Indeed, such a conclusion now requires the additional hypothesis that the reason for rejecting  $y$  (namely, the superiority of  $x$ ) is *perceived*.

But if the perceivability as well as the fact of preferences is to be debated, we can certainly devise a notation for encoding the former in the same way that the relation  $R$  encodes the latter. Given  $A$ , let us write  $R_A \subset R \cap [A \times A]$  for the relation containing the preferences that our agent perceives when faced with this menu. Allowing the choice problem to vary, we proceed to collect the associated perceived preference relations in a vector  $\mathbf{R} = \langle R_A \rangle_{A \in \mathcal{A}}$  to be called the decision maker’s *preference system*. This object will then be said to generate the choice function  $C$  when  $\forall A$  we have  $C(A) = \{x \in A : \forall y \in A \text{ we have } \neg[yR_A x]\}$ .

We shall refer to an arbitrary vector of relations on the sets in  $\mathcal{A}$  as a *relation system*. Such a vector will be said to exhibit a property normally ascribed to individual relations (e.g., acyclicity) when each of its components exhibits the property. And similarly, a relation system whose components each belong to a particular class will be designated as in the phrase “a system of partial orders.”

### 2.3. The requirement of nestedness

As the reader will no doubt have observed, every choice function is generated by a relation system and thus the assumption that behavior results from maximization of perceived preferences excludes no logical possibilities. There is, however, a quite natural — though by no means indisputable — restriction on preference systems that does constrain the choice functions they generate; one we shall refer to as the requirement of *nestedness*. To introduce this requirement, let us imagine that when confronted with a particular menu  $B \supset xy$  our decision maker perceives the preference  $xRy$ . When facing a different menu  $A \supset xy$  that is in some sense “no more complex” than  $B$ , we might then reasonably expect the decision maker again to perceive  $xRy$  on the grounds that only an increase in the complexity of the environment could have rendered it imperceptible. And although in the present abstract setting it is unclear how the complexity of a given choice problem is to be measured, we can treat the set inclusion relation as an indicator of *relative* complexity under the modest assumption that adding new alternatives to a problem cannot make it any simpler.

**Definition 1 (nestedness)** A relation system  $\mathbf{R}$  is said to be *nested* if  $\forall x, y \in A \subset B$  we have  $xR_B y$  only if  $xR_A y$ .<sup>3</sup>

One way to understand the content of the nestedness requirement is to try to imagine circumstances in which it might be violated.

1. Suppose that a convicted criminal can be executed either by hanging ( $x$ ) or by electrocution ( $y$ ), or can receive a Presidential pardon ( $z$ ). If the convict is allowed to choose his own fate, he may well view the larger choice problem  $xyz$  as simpler — in the sense

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<sup>3</sup>A precursor to this property appears in Anand [2, p. 339].

that it is easier to reach a decision — than the smaller problem  $xy$ . But while this scenario (envisioned by Yossi Feinberg) would then appear to violate the principle that complexity is aligned with set inclusion, it is in fact fully compatible with nestedness. Here adding the option of a pardon does not make it any easier to decide between the two modes of execution, it merely makes this comparison irrelevant.

2. In Sen’s [23, p. 753] thought experiment, a casual acquaintance will gladly accept rather than decline an invitation to stay for a cup of tea. When offered either tea or heroin, however, he is thought likely neither to take a refreshment nor to stay for further pleasantries. The failure of nestedness in this example is clear and uncontroversial. And as observed by Sen, the aberrant behavior here can be attributed to the “epistemic importance” of the heroin alternative, which merely by appearing on the menu can change the decision maker’s enthusiasm for tea in relation to a speedy exit.
3. Simonson and Tversky [26] investigate “extremeness aversion”; a preference for, e.g., medium-priced, medium-quality apples over apples whose price and quality are both simultaneously either high or low. In the same article, these authors also study “asymmetric dominance”; a preference for, e.g., good oranges over good bananas when bad oranges are available, and for good bananas over good oranges when bad bananas are available. Since neither the ordinal positions of alternatives in a ranking (on the one hand) nor preferences between good products induced by the presence of bad products (on the other) are stable under the addition or deletion of options from the menu, it is apparent that both of these phenomena can lead to violations of nestedness.<sup>4</sup>

The above examples are intended to clarify the nature of the nestedness hypothesis by suggesting arguments that might call this property into question. As we have seen, while the convict scenario describes behavior that is entirely consistent with the restriction, the behaviors imagined by Sen and observed by Simonson and Tversky are not. Moreover, the crucial question to be answered with respect to each of these putative counterexamples is the same: Can the addition of one alternative (such as, resp., a Presidential pardon, heroin, or bad bananas) cause the perception of a hitherto unperceived preference between two other alternatives (such as, resp., hanging and electrocution, tea and departure, or good oranges and good bananas). No violation of nestedness has yet been proposed that cannot be traced to this sort of causation. And thus it seems appropriate to conclude that *our theory is best suited for situations in which alternatives are evaluated in a pairwise manner* — though with a limited budget of cognitive resources available for making binary comparisons — and less well suited for situations in which menus are handled more holistically.<sup>5</sup>

#### 2.4. Expansion and contraction consistency

Having defined and critically (if briefly) evaluated the nestedness hypothesis, let us now proceed to examine its consequences for choice behavior. To this end, suppose that  $C$  is generated not by a single weak order  $R$ , as in Theorem 1, but rather by a nested system  $\mathbf{R}$  of weak orders. Given  $x, y \in A \cap B$  such that both  $x \in C(A)$  and  $y \in C(B)$ , suppose further that  $x \notin C(B)$ , in violation of the Weak Axiom. In this case there must exist a  $z \in B$

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<sup>4</sup>Gaertner and Xu [13] characterize choice of the median alternative, while Baigent and Gaertner [5] carry out a related exercise.

<sup>5</sup>Obviously, the same is true of theories based on the Weak Axiom, since any such theory involves a tacit assumption of nestedness (plus an assumption that cognitive resources are effectively unlimited).

such that both  $zR_Bx$  and  $\neg[zR_By]$ , and it then follows that  $yR_Bx$  (since  $\mathbf{R}$  is negatively transitive). But this rules out any possibility that  $A \subset B$ , as such an inclusion would imply that  $yR_Ax$  (since  $\mathbf{R}$  is nested), contradicting  $x \in C(A)$ .

The above reasoning establishes that when our decision maker's perceived preferences are assumed to be both nested and negatively transitive, violations of the Weak Axiom of a particular sort (namely, those for which  $A \subset B$ ) can be excluded. Precisely this restriction on the choice function has previously appeared, in the context of social choice theory, in the work of Bordes [6, p. 452] and of Sen [22, p. 66].

**Condition 2 (Strong Expansion)**  $\forall x, y \in A \subset B$  such that both  $x \in C(A)$  and  $y \in C(B)$ , we have  $x \in C(B)$ .

On the other hand, those violations of the Weak Axiom that *cannot* be ruled out are captured in the following condition generally credited to Chernoff [8, p. 429], but which also appears in Nash [18, p. 159].

**Condition 3 (Contraction)**  $\forall x \in A \subset B$  such that  $x \in C(B)$  we have  $x \in C(A)$ .

**Proposition 1 (Bordes, Sen)** *Contraction and Strong Expansion together are logically equivalent to the Weak Axiom.*

In Appendix A it is demonstrated that Strong Expansion is not only necessary for the choice function to be generated by a nested system of weak orders, this condition is also sufficient. That is to say, given any configuration of choice sets satisfying the condition, we can find a “revealed preference system” with the desired nestedness and ordering properties that could have been responsible for the decision maker's behavior. Together with Proposition 1, this fact enables us to present both the classical characterization theorem and our modification thereto in a way that accentuates their similarities.

**Theorem 2 [A]** *A choice function is generated by a weak order if and only if it satisfies Contraction and Strong Expansion. [B]* *A choice function is generated by a nested system of weak orders if and only if it satisfies Strong Expansion.*

And it is then apparent that permitting a classical decision maker to violate Contraction amounts to replacing the assumption of full perception with that of nestedness, while at the same time transferring the standard (weak) ordering properties from the preference relation to the preference system.

The following example illustrates the possibility of a choice function being generated by a nested system of weak orders but not by a single weak order; i.e., the “logical gap” between Theorems 2B and 1/2A.

**Example 1** The choice function taking  $wx \mapsto x$ ,  $wy \mapsto y$ ,  $wz \mapsto z$ ,  $xy \mapsto x$ ,  $xz \mapsto z$ ,  $yz \mapsto y$ ,  $wxy \mapsto xy$ ,  $wxz \mapsto xz$ ,  $wyz \mapsto yz$ ,  $xyz \mapsto xyz$ , and  $wxyz \mapsto xyz$  is generated by the nested system  $\mathbf{R}$  of weak orders containing the binary comparisons  $xR_{wx}w$ ,  $yR_{wy}w$ ,  $zR_{wz}w$ ,  $xR_{xy}y$ ,  $zR_{xz}x$ ,  $yR_{yz}z$ ,  $xR_{wxy}w$ ,  $yR_{wxy}w$ ,  $xR_{wxz}w$ ,  $zR_{wxz}w$ ,  $yR_{wyz}w$ ,  $zR_{wyz}w$ ,  $xR_{wxyz}w$ ,  $yR_{wxyz}w$ , and  $zR_{wxyz}w$ . Moreover, this function satisfies Strong Expansion and violates Contraction.

## 2.5. Foundations and the base relation

While Theorem 2B succeeds in characterizing choice behavior governed by a preference system  $\mathbf{R}$  exhibiting both nestedness and the standard ordering properties, this result places few restrictions on the relation  $R$  from which perceived preferences are drawn.<sup>6</sup> Indeed, the choice function in Example 1 satisfies Strong Expansion despite implying the preference cycle  $xRyRzRx$  (via the perceived preference cycle  $xR_{xy}yR_{yz}zR_{zx}x$ ). Yet we may wish to insist that our agent’s preferences themselves satisfy one or more consistency criteria quite apart from any question of cognition, since assumptions about his *powers* of perception (i.e., about  $\mathbf{R}$ ) are logically distinct from assumptions about the *objects* of this perception (i.e., about  $R$ ). And in fact a prohibition on preference cycles of the sort seen in Example 1 is precisely what we shall need in order to obtain (in Section 3) a numerical representation of the form of cognitively-constrained choice behavior under consideration.

We now formalize the notion of a relation system being “drawn from” a binary relation.

**Definition 2 (foundation)** A relation  $R$  is said to provide a *foundation* for a relation system  $\mathbf{R}$  if  $\bigcup_{A \in \mathcal{A}} R_A \subset R$ . Such a foundation is said to be *exact* if  $\bigcup_{A \in \mathcal{A}} R_A = R$ .

When perceived preferences are nested, an exact foundation for the preference system is always provided by the relation encoding binary choices.

**Definition 3 (base relation)** The *base relation*  $P^b$  is defined by  $xP^by$  if and only if  $y \notin C(xy)$  (i.e., both  $C(xy) = x$  and  $x \neq y$ ).

**Proposition 2** *If  $\mathbf{R}$  is a nested relation system that generates  $C$ , then  $P^b$  provides an exact foundation for  $\mathbf{R}$ .*

It follows that for the preference system to admit an acyclic foundation — any failure of which Herzberger [14, p. 195] deems “extremely pathological” — the base relation must itself be acyclic.

**Condition 4 (Base Acyclicity)** The relation  $P^b$  is acyclic.

And since, by Szpilrajn’s [27] Embedding Theorem, the transitive closure of any acyclic  $P^b$  can be strengthened to a weak (or even to a linear) order, this condition suffices to restore to the preference relation  $R$  the full complement of ordering properties originally demanded in Theorem 1.

**Theorem 3** *A choice function is generated by a nested system of weak orders that admits a weak order foundation if and only if it satisfies Base Acyclicity and Strong Expansion.*

The following example makes clear that Theorem 3 inhabits the logical space between Theorems 2B and 1/2A.

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<sup>6</sup>This can be attributed to the fact that expansion consistency conditions in general impose ordering properties on *perceived* preferences (as will become more apparent in Section 4), and it is only in the classical world of full perception — that is to say, under Contraction — that these properties are inherited by the preference assessments themselves.

**Example 2** The choice function taking  $wx \mapsto x$ ,  $wy \mapsto y$ ,  $wz \mapsto z$ ,  $xy \mapsto x$ ,  $xz \mapsto xz$ ,  $yz \mapsto y$ ,  $wxy \mapsto xy$ ,  $wxz \mapsto xz$ ,  $wyz \mapsto yz$ ,  $xyz \mapsto xyz$ , and  $wxyz \mapsto xyz$  is generated by the nested system  $\mathbf{R}$  of weak orders containing the comparisons  $xR_{wx}w$ ,  $yR_{wy}w$ ,  $zR_{wz}w$ ,  $xR_{xy}y$ ,  $yR_{yz}z$ ,  $xR_{wxy}w$ ,  $yR_{wxy}w$ ,  $xR_{wxz}w$ ,  $zR_{wxz}w$ ,  $yR_{wyz}w$ ,  $zR_{wyz}w$ ,  $xR_{wxyz}w$ ,  $yR_{wxyz}w$ , and  $zR_{wxyz}w$ ; which in turn admits the weak order foundation  $R$  containing the comparisons  $xRw$ ,  $yRw$ ,  $zRw$ ,  $xRy$ ,  $xRz$ , and  $yRz$ . Moreover, this function satisfies both Strong Expansion and (unlike the function in Example 1) Base Acyclicity, and violates Contraction.

### 3. Numerical representations

When the preference relation  $R$  can be encoded in a utility function  $f : X \rightarrow \mathfrak{R}$  in the sense that  $xRy$  if and only if  $f(x) > f(y)$ , and when in addition  $R$  generates  $C$ , it follows that  $\forall A$  we have  $C(A) = \{x \in A : f(x) \geq \max f[A]\}$ . In this case the members of each choice set are the available alternatives that meet the maximizing criterion (see Section 1), and when this is so we shall say that  $f$  constitutes a *maximizing representation* of the choice function. In characterizing this and other forms of numerical representation, it is convenient to adopt the simplifying assumption that the catalog  $X$  is finite. (See, e.g., Fishburn [12, p. 27] for a more general analysis.) And under this restriction one characterization is a straightforward consequence of Theorem 1.

**Theorem 4** *If  $X$  is finite, then a choice function admits a maximizing representation if and only if it satisfies the Weak Axiom.*

Since Proposition 1 and Theorem 4 together ensure (at least for the finite case) that Contraction is necessary for the existence of a maximizing representation, our replacement of this condition in Theorem 3 with the weaker Base Acyclicity axiom must introduce the possibility of behavior inconsistent with the utility-maximization model. A suitable generalization of this model (mentioned already in Section 1) posits together with the utility function  $f$  a *threshold mapping*  $\theta : \mathcal{A} \rightarrow \mathfrak{R}$  such that  $\forall A$  we have  $C(A) = \{x \in A : f(x) \geq \theta(A)\}$ . When in this sense the members of each choice set are the available alternatives that meet the satisficing criterion, the pair  $\langle f, \theta \rangle$  will be said to constitute a *satisficing representation* of the choice function.

As has just been suggested, a satisficing representation exists for any choice function generated by a nested system of weak orders that admits a weak order foundation. It is not true, however, that any choice function admitting a representation of this sort is thus generated.

**Example 3** The choice function defined by  $xy \mapsto xy$ ,  $xz \mapsto xz$ ,  $yz \mapsto z$ , and  $xyz \mapsto yz$  admits the satisficing representation  $\langle f, \theta \rangle$  with utility values  $f(x) = 0$ ,  $f(y) = 1$ , and  $f(z) = 2$  together with thresholds  $\theta(x) = \theta(xy) = \theta(xz) = 0$ ,  $\theta(y) = \theta(xyz) = 1$ , and  $\theta(z) = \theta(yz) = 2$ . Moreover, this function satisfies Base Acyclicity and violates both Strong Expansion and Contraction.

So to match the characterization in Theorem 3 exactly, it is necessary to identify the additional constraint on any satisficing representation imposed by the specified axioms.

Fix any representation  $\langle f, \theta \rangle$  of  $C$  and assume without loss of generality that  $\forall A$  we have  $\theta(A) = \min f[C(A)]$ . Now, given menus  $A \subset B$ , suppose that  $\max f[A] \geq \theta(B)$ . Choosing  $x, y \in C(A)$  such that  $f(x) = \min f[C(A)]$  and  $f(y) = \max f[C(A)]$ , we have



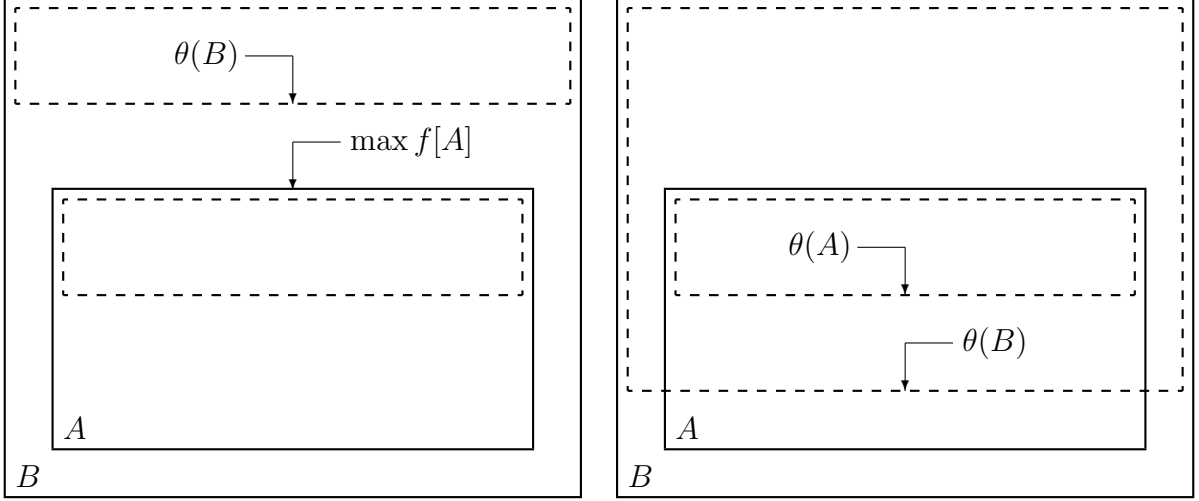


Figure 2: The property of expansiveness. Menus are indicated by solid boxes and choice sets by dashed boxes. The function  $f$  increases towards the top of the figure. A satisfying representation  $\langle f, \theta \rangle$  is expansive if  $\forall A \subset B$  we have either  $\max f[A] < \theta(B)$  (left panel) or  $\theta(A) \geq \theta(B)$  (right panel).

that  $f(x) = \theta(A)$  and  $f(y) \geq \theta(B)$  by assumption, and hence that  $y \in C(B)$ . If Strong Expansion holds, it then follows that  $x \in C(B)$  and therefore that  $\theta(A) = f(x) \geq \theta(B)$ .

**Definition 4 (expansiveness)** A satisfying representation  $\langle f, \theta \rangle$  is said to be *expansive* if  $\forall A \subset B$  such that  $\max f[A] \geq \theta(B)$  we have  $\theta(A) \geq \theta(B)$ .

This new property, depicted graphically in Figure 2, amounts to a restatement of Strong Expansion in terms of the representation at hand. (Specifically, the hypothesis  $\max f[A] \geq \theta(B)$  corresponds to  $\exists y \in A \cap C(B)$ , while the conclusion  $\theta(A) \geq \theta(B)$  corresponds to  $x \in C(A) \Rightarrow x \in C(B)$ .) Expansiveness is the constraint needed to bring satisficing behavior in line with Theorem 3, and imposing it yields a characterization that generalizes Theorem 4 in the same way that Theorem 2 generalizes Theorem 1.

**Theorem 5** *If  $X$  is finite, then: [A] A choice function admits a maximizing representation if and only if it satisfies Contraction and Strong Expansion. [B] A choice function admits an expansive satisficing representation if and only if it satisfies Base Acyclicity and Strong Expansion.*

As an exercise, the reader may wish to construct an expansive satisficing representation of the choice function in Example 2. (Note that another function belonging to the class characterized in Theorem 5B is exhibited in Figure 1 above.)

## 4. Alternative ordering assumptions

### 4.1. Acyclic orders

Our main results thus far (Theorems 2B and 3/5B) generalize the classical characterizations (Theorems 1/2A/4/5A) of choice behavior maximizing a preference relation with the properties of a weak order. While we have modified the assumption of maximization while

retaining the weak ordering hypothesis, other authors have done just the reverse. And by simultaneously modifying *both* of these assumptions, we can demonstrate the robustness of the link between nestedness and expansion consistency identified in Section 2.

We investigate first the case in which perceived preferences are required merely to be free of cycles.<sup>7</sup> Let  $C$  be generated by the nested system  $\mathbf{R}$  of acyclic orders, and suppose further that  $x \in \bigcap_{k \in K} A_k \subset \bigcup_{k \in K} A_k = B$  for a given index set  $K$  and associated collection of menus  $\langle A_k \rangle_{k \in K}$ . If  $x \notin C(B)$ , then  $\exists y \in B$  such that  $yR_Bx$ . It follows that  $\exists l \in K$  such that  $x, y \in A_l \subset B$ , and hence that  $yR_{A_l}x$  since  $\mathbf{R}$  is nested. But then  $x \notin C(A_l)$ , from which we can deduce that  $x \notin \bigcap_{k \in K} C(A_k)$  as well.

This reasoning establishes that whenever our decision maker's perceived preferences are nested, any intersection of choice sets is included in the choice set attached to the union of the corresponding menus. Once again, this is a condition that has been studied by Sen [20, p. 314].

**Condition 5 (Weak Expansion)**  $\forall x$  and  $\langle A_k \rangle_{k \in K}$  such that  $x \in \bigcap_{k \in K} C(A_k)$  we have  $x \in C(\bigcup_{k \in K} A_k)$ .

But while Sen uses this axiom together with Contraction to characterize behavior under full perception, we can use it on its own to obtain an analogous result for the cognitively-constrained case.

**Theorem 6 [A] (Sen)** *A choice function is generated by an acyclic order if and only if it satisfies Contraction and Weak Expansion. [B] A choice function is generated by a nested system of acyclic orders if and only if it satisfies Weak Expansion.*

The following example illustrates the logical gaps between Theorems 6B and 6A, on the one hand, and Theorems 6B and 2B, on the other.

**Example 4** The choice function defined by  $wx \mapsto w$ ,  $wy \mapsto y$ ,  $wz \mapsto wz$ ,  $xy \mapsto x$ ,  $xz \mapsto z$ ,  $yz \mapsto y$ ,  $wxy \mapsto x$ ,  $wxz \mapsto wxz$ ,  $wyz \mapsto yz$ ,  $xyz \mapsto y$ , and  $wxyz \mapsto xyz$  is generated by the nested system  $\mathbf{R}$  of acyclic orders containing the comparisons  $wR_{wx}x$ ,  $yR_{wy}w$ ,  $xR_{xy}y$ ,  $zR_{xz}x$ ,  $yR_{yz}z$ ,  $yR_{wxy}w$ ,  $xR_{wxy}y$ ,  $yR_{wyz}w$ ,  $zR_{xyz}x$ ,  $yR_{xyz}z$ , and  $yR_{wxyz}w$ . Moreover, this function satisfies Weak Expansion and violates both Contraction and Strong Expansion.

#### 4.2. Partial orders

Like acyclicity, transitivity offers “impressive credentials” ([14, p. 194]; see also Anand [2]) as a consistency criterion for preference assessments, and we now examine the impact of imposing both of these properties together. Let  $C$  be generated by the nested system  $\mathbf{R}$  of partial orders, and suppose further that  $x \notin C(A \cup y) \subset A$  for given  $x \in A$  and  $y \notin A$ . It follows that  $\exists z \in A \cup y$  such that  $zR_{A \cup y}x$ . If  $z \in A$ , then  $zR_Ax$  since  $\mathbf{R}$  is nested, and hence  $x \notin C(A)$ . Otherwise  $z = y$ , in which case  $z \notin C(A \cup y)$  and so  $\exists w \in A$  such that  $wR_{A \cup y}z$ . But then  $wR_{A \cup y}x$  since  $\mathbf{R}$  is transitive,  $wR_Ax$  since  $\mathbf{R}$  is nested, and again  $x \notin C(A)$ .

This reasoning establishes that whenever the preference system is both nested and transitive, adding an alternative to a menu cannot render a previously chooseable alternative unchooseable if the new alternative is itself unchooseable after its addition. What's more,

<sup>7</sup>This is the weakest ordering hypothesis of interest, as it is easily shown that a nested relation system can generate a choice function only if it is acyclic.

this remains true for any set  $(B \setminus A, \text{ where } A \subset B)$  of added alternatives, and it is in this more general form that we shall state the incremental expansion consistency condition corresponding to transitivity.<sup>8</sup> (Cf. [15, p. 58] “Aizerman’s Axiom.”)

**Condition 6 (Auxiliary Expansion)**  $\forall x \in A \subset B$  such that both  $x \in C(A)$  and  $C(B) \subset A$ , we have  $x \in C(B)$ .

**Theorem 7 [A]** *A choice function is generated by a partial order if and only if it satisfies Contraction, Weak Expansion, and Auxiliary Expansion. [B] A choice function is generated by a nested system of partial orders if and only if it satisfies Weak Expansion and Auxiliary Expansion.*

The following example illustrates the logical gaps between Theorems 7B and 7A, on the one hand, and Theorems 6B, 7B, and 2B, on the other.

**Example 5** The choice function defined by  $wx \mapsto w$ ,  $wy \mapsto y$ ,  $wz \mapsto w$ ,  $xy \mapsto x$ ,  $xz \mapsto x$ ,  $yz \mapsto yz$ ,  $wxy \mapsto xy$ ,  $wxz \mapsto w$ ,  $wyz \mapsto yz$ ,  $xyz \mapsto x$ , and  $wxyz \mapsto xy$  is generated by the nested system  $\mathbf{R}$  of partial orders containing the comparisons  $wR_{wx}x$ ,  $yR_{wy}w$ ,  $wR_{wz}z$ ,  $xR_{xy}y$ ,  $xR_{xz}z$ ,  $yR_{wy}w$ ,  $wR_{wx}x$ ,  $wR_{wz}z$ ,  $xR_{wx}z$ ,  $yR_{wyz}w$ ,  $xR_{xyz}y$ ,  $xR_{xyz}z$ ,  $yR_{wxyz}w$ , and  $xR_{wxyz}z$ . Moreover, this function satisfies both Weak Expansion and (unlike the function in Example 4) Auxiliary Expansion, and violates both Contraction and Strong Expansion.

#### 4.3. Linear orders

While only the weak order properties can be considered standard assumptions with regard to preference assessments, it is sometimes useful to adopt the stronger hypothesis that these judgments *linearly* order the available alternatives; to require, in other words, that the decision maker be able to affirm a definite preference between any two distinct options. To investigate this case, let  $C$  be generated by the nested system  $\mathbf{R}$  of linear orders, and suppose further that  $x, y \in C(A)$  for given  $x, y \in A$ . We then have both  $\neg[xR_A y]$  and  $\neg[yR_A x]$ , and it follows that  $x = y$  since  $\mathbf{R}$  is weakly connected.

This simple argument establishes that when our decision maker’s perceived preferences are weakly connected, each choice set contains precisely one element.

**Condition 7 (Univalence)**  $\forall x, y \in A$  such that  $x, y \in C(A)$  we have  $x = y$ .

It is well known that in the presence of this condition, Contraction and Strong Expansion are logically equivalent. And since both Strong Expansion and Univalence have been shown to hold under our current assumptions on the preference system, we can conclude that there is no distinction between full perception and cognitively-constrained choice in the linear order case.

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<sup>8</sup>When Weak Expansion is supplemented with Auxiliary Expansion, the consequences for the latent preference system are actually somewhat subtle. As Theorem 6B guarantees, Weak Expansion alone is sufficient for the choice function to be generated by a nested system  $\mathbf{R}$  of acyclic orders, and it follows that the transitive closure of  $\mathbf{R}$  is a system of *partial* orders that generates  $C$ . This fact fails to invalidate Theorem 7B, however, because the closure operation on relation systems does not in general preserve nestedness. In this sense our “intracomponent” (ordering) and “intercomponent” (nestedness) assumptions on the preference system do not act independently: Indeed, without nestedness no set of ordering properties considered thus far would enforce any restriction whatsoever on the choice function.

[a]	Contraction	
Strong Expansion and Univalence	linear order	
Strong Expansion	weak order	nested system of weak orders
Weak Expansion and Auxiliary Expansion	partial order	nested system of partial orders
Weak Expansion	acyclic order	nested system of acyclic orders

[b]	Strong Expansion	
Base Acyclicity	nested system of weak orders that admits a weak order foundation	
	nested system of weak orders	

[c]*	Strong Expansion	
Contraction	maximizing	
Base Acyclicity	expansive satisficing	

\*Here  $X$  is restricted to be finite.

Table I: Summary of characterization results. A choice function is generated by the indicated cognitive structure (Tables Ia–Ib) or admits a representation of the indicated type (Table Ic) if and only if it satisfies the marginal conditions.

**Theorem 8** [A] *A choice function is generated by a linear order if and only if it satisfies Strong Expansion and Univalence.* [B] *A choice function is generated by a nested system of linear orders if and only if it satisfies Strong Expansion and Univalence.*

The reason that we fail to obtain a generalization of the classical theory in the present instance can be expressed intuitively as follows. A decision maker whose preference system  $\mathbf{R}$  consists of weak orders is one who can always fully resolve his opinions at some level of precision, though his discriminatory capabilities may depend upon (the complexity of) the menu he faces. But if  $\mathbf{R}$  consists of *linear* orders then these capabilities *cannot* in fact depend upon the menu, since any weakly connected relation is by definition as discriminatory as can be. And with nestedness serving to link the preferences perceived in different choice problems, the decision maker turns out to be maximizing a single, menu-independent linear order of the classical sort.

#### 4.4. Summary of characterization results

Theorems 1–8, our characterization results for choice behavior based on fully or partially perceived preferences, are summarized in Table I. Here the row and column headings indicate different sets of conditions on the choice function, while the cells contain the associated cognitive structures or numerical representations. In Table Ia, juxtaposition of the two columns reveals that deleting Contraction from a received characterization corresponds to relaxing the assumption that the preference relation is fully perceived and assuming merely that the preference system is nested, while at the same time transferring any ordering properties from the one to the other.<sup>9</sup> (Of course, in the first row deleting Contraction has

<sup>9</sup>Note that these results link expansion consistency axioms to the (nestedness) requirement that perceived preferences be preserved under *contraction* of the menu of alternatives. Similarly, axioms of contraction consistency can be linked to the requirement that perceived preferences be preserved under *expansion* of the menu. Here the terminological inversion is a consequence of the inverse relationship between preference

no effect, since this condition is implied by the conjunction of Strong Expansion and Univalence.) In Table Ib, we see that reimposing Base Acyclicity after Contraction has been deleted corresponds to restoring to the preference relation the standard ordering properties. And in Table Ic (where  $X$  is restricted to be finite), we see that following this deletion and reimposition the decision maker exhibits a particular form of satisficing behavior.

## 5. Discussion

### 5.1. Related work

While it had seemed that the characterizations of cognitively-constrained choice behavior summarized in Table Ia were completely novel, a referee has drawn the author’s attention to an unpublished note by Duggan [10] that contains very similar results. Interestingly, Duggan reaches these results from the starting point of social (rather than individual) choice theory and uses them to draw conclusions about the properties of tournaments (rather than about satisficing).

Our findings about numerical representations summarized in Table Ic have their own antecedent in the work of Aleskerov and Monjardet [1] (for which citation the author is indebted to Monjardet himself). From the present perspective, the contribution of these authors is to have characterized satisficing behavior per se, absent both the expansiveness requirement and the underlying motivation in terms of preference and cognition. This they have done using an acyclicity condition somewhat stronger than that introduced in Section 2.5 above.

**Definition 5 (separation relation)** *The separation relation  $P^s$  is defined by  $xP^sy$  if and only if  $\exists A$  such that both  $x \in C(A)$  and  $y \in A \setminus C(A)$ .*

**Condition 8 (Separation Acyclicity)** *The relation  $P^s$  is acyclic.*

**Theorem 9 (Aleskerov and Monjardet)** *If  $X$  is finite, then a choice function admits a satisficing representation if and only if it satisfies Separation Acyclicity.*

Manzini and Mariotti [17] consider choice functions produced lexicographically from finite sequences of binary relations (which need not be weak orders as in the standard lexicographic model). In general such functions need bear no particular relationship to those studied in the present paper. But when the sequence of relations is of length two (the case of “shortlisting”), the choice function generated can be shown to satisfy Weak Expansion and therefore to fall under the purview of our Theorem 6B.

The model of “rationalization by multiple rationales” proposed by Kalai et al. [16] (see also Apesteguia and Ballester [3]) can likewise be translated into the language of relation systems, and nestedness can be guaranteed by adding fairly natural assumptions to their framework.

### 5.2. Idempotence and reductionism

Consider again the choice function displayed in Figure 1. When facing the menu  $wyz$ , a decision maker exhibiting this behavior succeeds in ruling out alternative  $w$  but remains

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and choice: A (perceived) preference for one alternative over another is a reason *not* to choose the second alternative, but not in itself a reason to choose the first. (See Sen [22, p. 66].)

undecided between  $y$  and  $z$ . Since this leaves him with the new menu  $yz$ , and since the associated choice set contains only  $z$ , should we not require that  $wyz \mapsto z$  in the first place? Note that this amounts to a demand that the following condition be satisfied.

**Condition 9 (Idempotence)**  $\forall x \in A$  such that  $x \in C(A)$  we have  $x \in C(C(A))$ .

Figuratively speaking, the reason we do not insist on Idempotence is that by the time our decision maker gets around to the second application of his choice function to the “menu” in his hand, *the waiter has already reappeared*. The agent facing  $wyz$  in Figure 1 is certain that he does not wish to consume wildebeest, but has not yet been able to come to a definite conclusion about the relative culinary merits of yak and zebra. He is dining in a large group, however, his companions are already placing their orders, and he will next be called upon to do so. Allowing the agent at this point to apply  $C$  a second time would amount to delaying the ordering process unconscionably, and politeness dictates that in this situation he should make a selection at once despite not yet having perceived his true preference between yak and zebra. Less fancifully, we can say that Idempotence is not an attractive condition because it runs directly counter to our basic assumption that the decision maker is cognitively constrained: Indeed, the assertion that spending his cognitive budget once is no different than spending it twice would seem to imply that this budget is either vanishing or infinite.

The above argument against adopting Idempotence raises several related questions: If our agent has a fixed cognitive budget, where is his budget constraint? If his perceptual capabilities are limited, what are the nature and origin of the supposed limitations? And if the complexity of the decision-making environment prevents the construction of one preference while allowing that of another, what explains this differential treatment?

Questions such as these implicitly advocate further reductionism; the derivation of the decision maker’s preference system from something more concrete and fundamental. But while there would be nothing wrong in principle with attempting to base our theory on a more elaborate cognitive model, a number of good reasons can be given for avoiding this approach at present. Firstly, theories that seek to express bounded rationality directly as optimization under (cognitive or other) constraints often lead to conceptual difficulties such as those discussed by Conlisk [9, pp. 686 ff.] and Elster [11, p. 25]. Secondly, detailed hypotheses about the process of decision making are most sensibly formulated in the context of particular choice scenarios, such as the selection of an investment portfolio, but specializing the analysis to any one application of this sort would tend to obscure the generality of our contribution. And thirdly, an answer to the question of why one preference is perceived while another is not has no counterpart in the theory of maximizing behavior, and so any attempt to address this issue would cause the parallel development of our theory and its classical progenitor to break down.

To appreciate this last point, imagine challenging the significance of the classical Theorems 1 and 4 by posing questions similar to those above: If our agent prefers one alternative over another, what reasons can he give for feeling this way? How did he go about evaluating the alternatives and which of their characteristics weighed heavily in the comparison? To what extent is the stated preference a product of introspection and to what extent one of tradition, fashion, or other social factors?

These may be interesting questions, and answering them essential for understanding the decision maker’s behavior completely. But doing so is not the task of classical choice theory,

just as seeking to explain the mechanisms of cognition is not our aim here. In the same way that standard results take as given the preference relation specifying tastes and attitudes, we take as given the preference *system* specifying cognitive capabilities. Thus, in the new theory as in the old, the focus is not on the origins of preference and perception but rather on their implications for and revelation by observed choices.

## Appendix

### A. Revelation of perceived preference

Given a relation  $R$ , let us write  $R\uparrow(A) = \{x \in A : \forall y \in A \text{ we have } \neg[yRx]\}$  for the set of  $R$ -maximal alternatives on the menu  $A$ . Given a relation system  $\mathbf{R}$ , we similarly define the set  $\mathbf{R}\uparrow(A) = \{x \in A : \forall y \in A \text{ we have } \neg[yR_Ax]\}$ . The expression  $C = R\uparrow$  (resp.,  $C = \mathbf{R}\uparrow$ ) will then indicate that  $R$  (resp.,  $\mathbf{R}$ ) generates  $C$ , an assertion that can be usefully factored — adopting a straightforward notation — into the *upper bound* inclusion  $C \subset R\uparrow$  (resp.,  $C \subset \mathbf{R}\uparrow$ ) and the *lower bound* inclusion  $R\uparrow \subset C$  (resp.,  $\mathbf{R}\uparrow \subset C$ ). And finally, given two relation systems  $\mathbf{R}$  and  $\mathbf{Q}$ , we shall write  $\mathbf{R} \subset \mathbf{Q}$  when  $\forall A$  we have  $R_A \subset Q_A$ .

Our first task is to define a revealed preference relation and an analogous relation system that can be used to demonstrate sufficiency of the stated axioms in the characterization results of Sections 2 and 4.

**Definition 6** [A] (**global relation**) The *global relation*  $P^g$  is defined by  $xP^gy$  if and only if  $\forall A$  such that  $x \in A$  we have  $y \notin C(A)$ . [B] (**local relation system**) The *local relation system*  $\mathbf{P}^l$  is defined by  $xP_B^ly$  if and only if  $\forall A \subset B$  such that  $x \in A$  we have  $y \notin C(A)$ .

The following result collects a number of useful facts about these objects. (Recall that  $xP^by$  if and only if  $y \notin C(xy)$ , while  $xP^sy$  if and only if  $\exists A$  such that both  $x \in C(A)$  and  $y \in A \setminus C(A)$ .)

**Proposition 3** [A]  $P_X^l = P^g \subset P^b \subset P^s$ . [B]  $P^b$  provides an exact foundation for  $\mathbf{P}^l$ . [C]  $\mathbf{P}^l$  is nested. [D]  $C \subset \mathbf{P}^l\uparrow \subset P^g\uparrow$ . [E] Contraction implies that  $P^b \subset P^g$ , while Strong Expansion implies that  $P^s \subset P^b$ . [F] Given any relation  $R$  such that  $C \subset R\uparrow$ , we have  $R \subset P^g$ . [G] Given any nested relation system  $\mathbf{R}$  such that  $C \subset \mathbf{R}\uparrow$ , we have  $\mathbf{R} \subset \mathbf{P}^l$ .

Since the upper bound inclusions  $C \subset \mathbf{P}^l\uparrow \subset P^g\uparrow$  hold tautologically, the following conditions demarcate the classes of choice functions generated by, respectively, the global relation and the local relation system.

**Condition 10** [A] (**Global Lower Bound**)  $P^g\uparrow \subset C$ . [B] (**Local Lower Bound**)  $\mathbf{P}^l\uparrow \subset C$ .

Moreover, when  $C$  is generated by *any* relation  $R$  (resp., nested relation system  $\mathbf{R}$ ), then Proposition 3F (resp., 3G) implies that  $P^g\uparrow \subset R\uparrow = C$  (resp.,  $\mathbf{P}^l\uparrow \subset \mathbf{R}\uparrow = C$ ). Thus we can conclude that Global and Local Lower Bound provide exact characterizations of choice behavior guided by, respectively, fully perceived preference relations and nested preference systems.

The job of linking the above lower bound conditions with the more intuitive expansion and contraction consistency axioms used in the main text is carried out by the next result.

**Proposition 4** [A] *Contraction and Weak Expansion together are logically equivalent to Global Lower Bound.* [B] *Weak Expansion is logically equivalent to Local Lower Bound.*

And it then remains only to confirm that under the relevant conditions, the local relation system (and hence, by Proposition 3A, also the global relation) exhibits the ordering properties called for in our various characterizations.

**Proposition 5** [A]  $\mathbf{P}^1$  *is a system of acyclic orders.* [B] *Auxiliary Expansion implies that  $\mathbf{P}^1$  is a system of partial orders.* [C] *Strong Expansion implies that  $\mathbf{P}^1$  is a system of weak orders.* [D] *Strong Expansion and Univalence together imply that  $\mathbf{P}^1$  is a system of linear orders.* [E] *Base Acyclicity implies that  $\mathbf{P}^1$  admits a weak order foundation.*

## B. Logical implications among conditions

Several logical implications among the choice-theoretic axioms defined in this paper have not yet been mentioned, but are used in the proofs of (or otherwise pertinent to) our characterizations. For completeness, these facts are stated in the following result.

**Proposition 6** [A] *Contraction implies Base Acyclicity.* [B] *Strong Expansion implies both Weak Expansion and Auxiliary Expansion.* [C] *Separation Acyclicity implies Base Acyclicity.* [D] *Base Acyclicity and Strong Expansion together imply Separation Acyclicity.*

All relevant relationships among our various axioms are summarized in Figure 3.

## C. Selected proofs

Here, as in Figure 3, conditions on the choice function are indicated by their initials. Moreover, Theorem 1 is referred to as T1, Proposition 1 as P1, and so on.

**Proof of Theorem 2.** [B] Let SE hold, in which case WE holds by P6B. Then  $\mathbf{P}^1$  is a nested system of weak orders that generates  $C$  by P3C–D, P4B, and P5C. || Let  $C$  be generated by a nested system  $\mathbf{R}$  of weak orders. If SE fails, then  $\exists x, y \in A \subset B$  such that  $x \in C(B) = \mathbf{R}\uparrow(B)$  and  $y \in C(A) \setminus C(B) = \mathbf{R}\uparrow(A) \setminus \mathbf{R}\uparrow(B)$ . It follows that  $\neg[yR_Bx]$ ,  $\neg[xR_Ay]$ , and thus  $\neg[xR_By]$  since  $\mathbf{R}$  is nested. Moreover,  $\exists z \in B$  such that  $zR_By$ , and hence  $zR_Bx$  since  $\mathbf{R}$  is negatively transitive. But then  $x \notin C(B)$ , contradicting  $x \in C(B)$ . Therefore SE holds.

**Proof of Proposition 2.** Let  $\mathbf{R}$  be nested and let  $C = \mathbf{R}\uparrow$ . We then have  $xP^b y$  if and only if  $xR_{xy}y$ , and it follows that  $P^b = \bigcup_{x,y \in X} R_{xy} \subset \bigcup_{A \in \mathcal{A}} R_A$ . On the other hand, since  $\mathbf{R}$  is nested we have  $xR_Ay$  only if  $xR_{xy}y$  and therefore only if  $xP^b y$ , and it follows that  $\bigcup_{A \in \mathcal{A}} R_A \subset P^b$ .

**Proof of Theorem 3.** Let both BA and SE hold, in which case WE holds by P6B. Then  $\mathbf{P}^1$  is a nested system of weak orders that both admits a weak order foundation and generates  $C$  by P3C–D, P4B, P5C, and P5E. || Let  $C$  be generated by a nested system  $\mathbf{R}$  of weak orders that admits a weak order foundation  $R$ , in which case SE holds by T2B. Since  $P^b$  provides an exact foundation for  $\mathbf{R}$  by P2, it follows that  $P^b = \bigcup_{A \in \mathcal{A}} R_A \subset R$ . But then, since  $R$  is acyclic,  $P^b$  too must be acyclic. Therefore BA holds.



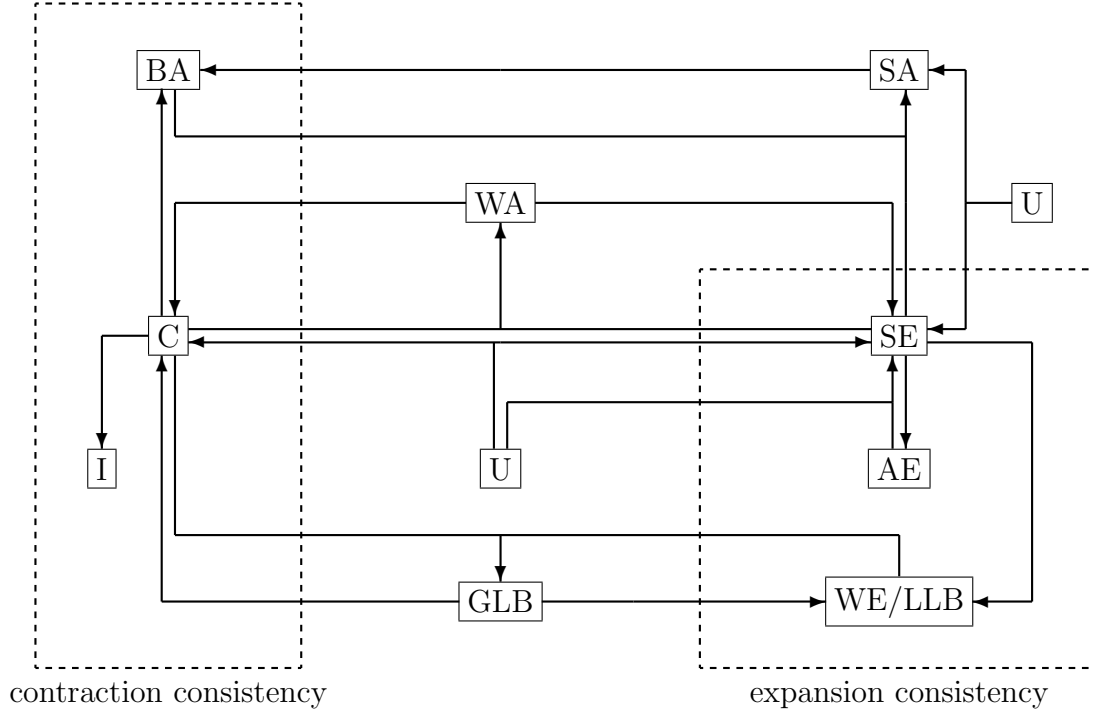


Figure 3: Logical implications among conditions. A condition on the choice function is indicated by its initials (e.g., the Weak Axiom by WA) and an implication by a directed edge (e.g., WA implies C). Intersecting edges indicate a joint hypothesis (e.g., C and SE together imply WA).

**Proof of Theorem 5.** [B] Let  $X$  be finite and let both BA and SE hold, in which case SA holds by P6D and  $C$  admits a satisficing representation  $\langle f, \theta \rangle$  by T9. Without loss of generality, we may assume that  $\forall A$  we have  $\theta(A) = \min f[C(A)]$ . Given  $A \subset B$  such that  $\max f[A] \geq \theta(B)$ , we have that  $\exists y \in A \cap C(B)$  and hence  $C(A) \subset C(B)$  by SE. It then follows that  $\theta(A) = \min f[C(A)] \geq \min f[C(B)] = \theta(B)$ , and therefore  $\langle f, \theta \rangle$  is expansive. || Let  $X$  be finite and let  $C$  admit an expansive satisficing representation  $\langle f, \theta \rangle$ , in which case SA holds by T9 and BA holds by P6C. Given  $x, y \in A \subset B$  such that both  $x \in C(A)$  and  $y \in C(B)$ , we have that  $\max f[A] \geq f(y) \geq \theta(B)$ , that  $f(x) \geq \theta(A) \geq \theta(B)$  since  $\langle f, \theta \rangle$  is expansive, and hence that  $x \in C(B)$ . Therefore SE holds.

**Proof of Theorem 6.** [B] Let WE hold. Then  $\mathbf{P}^1$  is a nested system of acyclic orders that generates  $C$  by P3C–D, P4B, and P5A. || Let  $C$  be generated by a nested system  $\mathbf{R}$  of acyclic orders. Then  $\mathbf{R} \subset \mathbf{P}^1$  by P3G, and hence  $\mathbf{P}^1 \uparrow \subset \mathbf{R} \uparrow = C$ . But then WE holds by P4B.

**Proof of Theorem 7.** [B] Let both WE and AE hold. Then  $\mathbf{P}^1$  is a nested system of partial orders that generates  $C$  by P3C–D, P4B, and P5B. || Let  $C$  be generated by a nested system  $\mathbf{R}$  of partial orders, in which case WE holds by T6B. If AE fails, then  $\exists x \in A \subset B$  such that  $C(B) \subset A$  and  $x \in C(A) \setminus C(B) = \mathbf{R} \uparrow(A) \setminus \mathbf{R} \uparrow(B)$ , and it follows that  $\exists y_1 \in B$  such that  $y_1 \mathbf{R}_B x$ . [*Inductive step begins.*] Let  $y_k \in B$  be such that  $y_k \mathbf{R}_B x$ . If  $y_k \in A$  then  $y_k \mathbf{R}_A x$  since  $\mathbf{R}$  is nested, contradicting  $x \in \mathbf{R} \uparrow(A)$ . Thus  $y_k \in B \setminus A \subset B \setminus C(B)$ , and so  $y_k \notin C(B) = \mathbf{R} \uparrow(B)$  and  $\exists y_{k+1} \in B$  such that  $y_{k+1} \mathbf{R}_B y_k \mathbf{R}_B x$ .

Furthermore,  $y_{k+1}R_Bx$ , since  $\mathbf{R}$  is transitive. [*Inductive step ends.*] Using induction, we can construct a set  $D = y_1y_2 \cdots \subset B$  with the property that  $\forall k \geq 1$  we have  $y_{k+1}R_By_k$  and hence  $y_{k+1}R_Dy_k$  since  $\mathbf{R}$  is nested. But then  $C(D) = \mathbf{R}\uparrow(D) = \emptyset$ , contradicting  $C(D) \neq \emptyset$ . Therefore AE holds.

**Proof of Theorem 8.** [B] Let both SE and U hold, in which case WE holds by P6B. Then  $\mathbf{P}^1$  is a nested system of linear orders that generates  $C$  by P3C–D, P4B, and P5D. || Let  $C$  be generated by a nested system  $\mathbf{R}$  of linear orders, in which case SE holds by T2B. If U fails, then  $\exists x, y \in A$  such that  $x \neq y$  and  $x, y \in C(A) = \mathbf{R}\uparrow(A)$ . But then  $\neg[xR_Ay]$  and  $\neg[yR_Ax]$ , contradicting the weak connectedness of  $\mathbf{R}$ . Therefore U holds.

**Proof of Proposition 3.** [A] That  $\mathbf{P}_X^1 = \mathbf{P}^g$  is immediate. || If  $y\mathbf{P}^g x$ , then  $x \notin C(xy)$  and hence  $y\mathbf{P}^b x$ . || If  $x\mathbf{P}^b y$  then  $y \notin C(xy)$  and  $x \in C(xy)$ , so  $x\mathbf{P}^g y$ . [B] Since  $x\mathbf{P}^b y$  if and only if  $x\mathbf{P}_{xy}^1 y$ , we have  $\mathbf{P}^b = \bigcup_{x,y \in X} \mathbf{P}_{xy}^1 \subset \bigcup_{A \in \mathcal{A}} \mathbf{P}_A^1$ . On the other hand, since  $\mathbf{P}^1$  is nested we have  $x\mathbf{P}_A^1 y$  only if  $x\mathbf{P}_{xy}^1 y$  and hence only if  $x\mathbf{P}^b y$ , and it follows that  $\bigcup_{A \in \mathcal{A}} \mathbf{P}_A^1 \subset \mathbf{P}^b$ . [C] That  $\mathbf{P}^1$  is nested is immediate. [D] Given  $x, y \in A$ , if  $y\mathbf{P}_A^1 x$  then  $x \notin C(A)$ . By contraposition,  $C \subset \mathbf{P}^1\uparrow$ . || Given  $A$ , we have  $\mathbf{P}^g \cap (A \times A) = \mathbf{P}_X^1 \cap (A \times A) \subset \mathbf{P}_A^1$  since  $\mathbf{P}^1$  is nested, and it follows that  $\mathbf{P}_A^1\uparrow(A) \subset \mathbf{P}^g\uparrow(A)$ . Therefore  $\mathbf{P}^1\uparrow \subset \mathbf{P}^g\uparrow$ . [E] Let C hold. If  $x\mathbf{P}^b y$  then  $y \notin C(xy)$  and  $\forall A \supset xy$  we have  $y \notin C(A)$  by C, which is to say that  $x\mathbf{P}^g y$ . Therefore  $\mathbf{P}^b \subset \mathbf{P}^g$ . || Let SE hold. If  $x\mathbf{P}^s y$  then  $\exists A$  such that  $x \in C(A)$  and  $y \in A \setminus C(A)$ . Since  $xy \subset A$  we then have  $y \notin C(xy)$  by SE, which is to say that  $x\mathbf{P}^b y$ . Therefore  $\mathbf{P}^s \subset \mathbf{P}^b$ . [F] Let  $C \subset \mathbf{R}\uparrow$ . If  $\neg[x\mathbf{P}^g y]$  then  $\exists A$  such that  $x \in A$  and  $y \in C(A) \subset \mathbf{R}\uparrow(A)$ , and thus  $\neg[xRy]$ . By contraposition,  $\mathbf{R} \subset \mathbf{P}^g$ . [G] Let  $\mathbf{R}$  be nested and let  $C \subset \mathbf{R}\uparrow$ . If  $\neg[x\mathbf{P}_A^1 y]$  then  $\exists B \subset A$  such that  $x \in B$  and  $y \in C(B) \subset \mathbf{R}\uparrow(B)$ . But then  $\neg[xR_By]$  and hence  $\neg[xR_Ay]$  since  $\mathbf{R}$  is nested. By contraposition,  $\mathbf{R} \subset \mathbf{P}^1$ .

**Proof of Proposition 4.** [B] Let WE hold. Given  $x, y \in B$  such that  $x \in \mathbf{P}^1\uparrow(B)$ , we have that  $\neg[y\mathbf{P}_B^1 x]$  and so  $\exists A_y \subset B$  such that  $y \in A_y$  and  $x \in C(A_y)$ . It follows that  $x \in \bigcap_{y \in B} C(A_y) \subset C(\bigcup_{y \in B} A_y) = C(B)$  by WE, and therefore LLB holds. || Let LLB hold. If  $x \in \bigcap_{k \in K} C(A_k)$  then  $x \in \bigcap_{k \in K} \mathbf{P}^1\uparrow(A_k)$  by P3D,  $x \in \mathbf{P}^1\uparrow(\bigcup_{k \in K} A_k)$  since  $\mathbf{P}^1$  is nested, and  $x \in C(\bigcup_{k \in K} A_k)$  by LLB. Therefore WE holds.

**Proof of Proposition 5.** [A] If  $\mathbf{P}^1$  is not acyclic, then  $\exists x_1, x_2, \dots, x_n \in B$  such that  $x_1\mathbf{P}_B^1 x_2\mathbf{P}_B^1 \cdots \mathbf{P}_B^1 x_n\mathbf{P}_B^1 x_1$ . Letting  $A = x_1x_2 \cdots x_n \subset B$ , we have that  $x_1\mathbf{P}_A^1 x_2\mathbf{P}_A^1 \cdots \mathbf{P}_A^1 x_n\mathbf{P}_A^1 x_1$  since  $\mathbf{P}^1$  is nested. But then  $\emptyset = \mathbf{P}^1\uparrow(A) \supset C(A)$  by P3D, contradicting  $C(A) \neq \emptyset$ . Therefore  $\mathbf{P}^1$  is a system of acyclic orders. [B] Let AE hold. Then  $\mathbf{P}^1$  is acyclic by P5A. If  $\mathbf{P}^1$  is not transitive, then  $\exists x, y, z \in D$  such that  $x\mathbf{P}_D^1 y\mathbf{P}_D^1 z$  and  $\neg[x\mathbf{P}_D^1 z]$ . In this case  $\exists A \subset D$  such that  $x \in A$  and  $z \in C(A)$ , and letting  $B = A \cup y \subset D$  we have that  $x\mathbf{P}_B^1 y\mathbf{P}_B^1 z$  since  $\mathbf{P}^1$  is nested. Since  $C(B) \subset \mathbf{P}^1\uparrow(B)$  by P3D it follows that  $C(B) \subset A \subset B$  and  $z \in C(A) \setminus C(B)$ , contradicting AE. Therefore  $\mathbf{P}^1$  is a system of partial orders. [C] Let SE hold. Then  $\mathbf{P}^1$  is acyclic and transitive by P5A–B and P6B. If  $\mathbf{P}^1$  is not negatively transitive, then  $\exists x, y, z \in D$  such that  $\neg[x\mathbf{P}_D^1 y]$ ,  $\neg[y\mathbf{P}_D^1 z]$ , and  $x\mathbf{P}_D^1 z$ . In this case  $\exists A \subset D$  such that  $x \in A$  and  $y \in C(A)$  and, moreover,  $\exists B \subset D$  such that  $y \in B$  and  $z \in C(B)$ . In view of the fact that  $x, z \in A \cup B \subset D$ , we have also  $x\mathbf{P}_{A \cup B}^1 z$  since  $\mathbf{P}^1$  is nested and hence  $z \notin C(A \cup B)$  by P3D. Now, given any  $w \in C(A \cup B)$ , if either  $w \in B$  or  $y \in C(A \cup B)$  then both  $C(A \cup B) \cap B \neq \emptyset$  and  $z \in C(B) \setminus C(A \cup B)$ , contradicting SE. Alternatively, if both  $w \notin B$  and  $y \notin C(A \cup B)$  then both  $w \in C(A \cup B) \cap A \neq \emptyset$  and  $y \in C(A) \setminus C(A \cup B)$ , again contradicting SE. Therefore  $\mathbf{P}^1$  is a system of weak orders. [D] Let both SE and U hold.

Then  $\mathbf{P}^1$  is acyclic, transitive, and negatively transitive by P5A–C and P6B. If  $\mathbf{P}^1$  is not weakly connected, then  $\exists x, y \in D$  such that  $x \neq y$ ,  $\neg[xP_D^1 y]$ , and  $\neg[yP_D^1 x]$ . In this case  $\exists A \subset D$  such that  $x \in A$  and  $y \in C(A)$  and, moreover,  $\exists B \subset D$  such that  $y \in B$  and  $x \in C(B)$ . If  $x \in C(xy)$ , then  $x \in C(A)$  by SE, contradicting U. Alternatively, if  $y \in C(xy)$  then  $y \in C(B)$  by SE, again contradicting U. Therefore  $\mathbf{P}^1$  is a system of linear orders. [E] Let BA hold. Then the transitive closure of  $P^b$  is a partial order which by the Embedding Theorem can be strengthened to a weak order R. And since  $P^b$  provides a foundation for  $\mathbf{P}^1$  by P3B, the weak order  $R \supset P^b$  also provides a foundation for  $\mathbf{P}^1$ .

**Proof of Proposition 6.** [A] Let C hold. Since  $P^b \subset P^s$  by P3E and  $P^s$  is acyclic by P3A and P5A,  $P^b$  too is acyclic and BA holds. [B] Let SE hold. Given  $x \in \bigcap_{k \in K} C(A_k)$  and  $y \in C(\bigcup_{k \in K} A_k)$ , we have that  $\exists l \in K$  such that  $y \in A_l$ . We then have both  $A_l \subset \bigcup_{k \in K} A_k$  and  $x \in C(A_l)$ , and it follows that  $x \in C(\bigcup_{k \in K} A_k)$  by SE. Hence WE holds. || Let SE hold. Given  $x \in A \subset B$  such that both  $x \in C(A)$  and  $C(B) \subset A$ , we have that  $\exists y \in C(B) \cap A$  and so  $x \in C(B)$  by SE. Therefore AE holds. [C] Let SA hold. Since  $P^b \subset P^s$  by P3A and  $P^s$  is acyclic,  $P^b$  too is acyclic and BA holds. [D] Let both BA and SE hold. Since  $P^s \subset P^b$  by P3E and  $P^b$  is acyclic,  $P^s$  too is acyclic and SA holds.

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